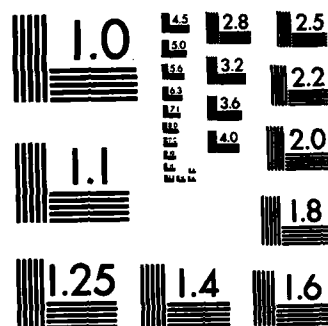MICROCOPY RESOLUTION TEST CHART

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BIB DESIGNS WITH BLOCKS OF

MAXIMAL MULTIPLICITY*

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ABSTRACT

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as its support. We construct a BIB design with parameters
$v(\geq 7)$, $k = 3$, $\lambda = v - 2$ which contains a block of
maximal multiplicity and has support $\binom{v}{3} - 4(v-3)$. Any
BIB design which contains such a block and has parameters
as above must be supported on at most $\binom{v}{3} - 4(v-3)$ blocks.

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support, block of maximal multiplicity, trades, finite
population sampling.

1. Introduction

In the course of performing an experiment using a block design, a cost is often attached to each block. It then becomes of interest to preserve only those symmetries of the design which are of decisive statistical interest with the object of somehow reducing the cost. It has been noted, along these lines (see Wynn (1977) or Foody and Hedayat (1977)), that of primary interest is only the property that each pair of distinct varieties appear in the same (or nearly same) number of blocks. One is then naturally tempted to select more copies of a least expensive block, so long as the above mentioned property on pairs is preserved, and thus reduce the cost without any loss of information.

The general problem of making such selections is of considerable difficulty. We shall investigate here the complete design on $v(\geq 7)$ varieties and blocks of size 3, with the assumption that a block is considerably less costly than the rest and the objective of selecting this block a maximal number of times without altering the original pair inclusions of varieties. BIB designs with repeated blocks have other applications in designs of experiments and survey sampling as has been pointed out in Foody and Hedayat (1977), Hedayat (1979) and Wynn (1977). The set of distinct blocks of a block design is known as the support of the design. We shall denote by b^* the number of blocks in the support. A BIB design on v varieties and b blocks of size k, of which b^* are in the support, in which a pair of distinct

varieties appears in λ blocks and each variety appears in r blocks, will be denoted by $BIB(v,k,\lambda|b^*)$. It is easy to count that

$$b = \frac{v(v-1)}{k(k-1)} \lambda \quad \text{and} \quad r = \frac{bk}{v} .$$

If we denote by f_i the frequency of each block in the support, then in a $BIB(v,k,\lambda|b^*)$ we clearly have $0 < f_i \leq \lambda$. There are cases in which this upper bound can never be achieved. Mann(1969) showed that also $f_i \leq bv^{-1}$. To summarize, then

$$\max_i f_i \leq \lambda \quad \text{if} \quad v \geq k(k-1) + 1$$

and

$$\max_i f_i \leq bv^{-1} \quad \text{if} \quad v < k(k-1) + 1$$

If we write $b \geq f_i v$ and sum over i in the support of the BIB design, we obtain $b^* \geq v$; which is a generalization of Fisher's inequality.

In the case of v varieties $(v \geq 7)$ and blocks of size 3 we shall work with the complete design and show how to produce a BIB design with the same parameters as the complete design, but of smaller support and with a block repeated a maximal number of times.

2. On trades

Let $P = \{1,2,\ldots,v\}$ be the set of varieites. For a positive integer $1 \le j \le v$, we denote by $v\}j$ the collection of all subsets of j varieties out of the v. Hedayat and Li (1979, 1980) introduced trades to reduce the support of various designs. The trades are useful and can be interpreted in a number of ways: as vectors in the kernel of the adjacency matrix between t-subsets and blocks ($t < k =$ block size), as triangulated surfaces or as special kinds of homogeneous polynomials. For this latter interpretation see Graham, Li and Li (1980).

We remind the reader of what a trade is. A collection, T, of $2h$ elements of $v\}k$ is said to form a (v,k,t) trade of volume h if we can partition T into two non-overlapping subcollections T_1 and T_2 of size h each such that each element of $v\}t$ is contained in the same number of elements of both T_1 and T_2. That is to say, T_1 and T_2 have the same covering properties for the elements of $v\}t$. (Note that this definition forces $k > t$.)

As an example, let $P = \{1,2,\ldots,6\}$, $k = 3$ and $t = 2$. Then the following T_1 and T_2 form a $(6,3,2)$ trade of volume 4.

$\underline{T_1}$	$\underline{T_2}$
1 2 3	1 3 4
3 4 5	1 2 6
2 5 6	2 3 5
1 4 6	4 5 6

From the definition of a (v,k,t) trade it is obvious that
if T is a (v,k,t) trade it is also a (v',k,t) trade for
any v' ≥ v . When k = 3 , we can, for convenience, repre-
sent the v⌡3 blocks by equilateral triangles whose vertices
are labeled by the varieites of the respective block. With
this representation of blocks of size 3 we can color the
blocks of T_1 white and those of T_2 black in a (v,3,2)
trade and thus achieve a geometrical representation of a
trade as is done in Hedayat and Li (1980).

It is clear that if a BIB design contains in its support
the part T_1 of a trade T with nonoverlapping subcollec-
tions T_1 and T_2 , then by replacing the blocks in T_1 by
those in T_2 we again obtain a BIB design. The support and
support size, however, are likely to have changed in the
process. We say that we added the trade T to the design.

By using trades, Foody and Hedayat (1977), Hedayat and
Li (1979, 1980) and Hedayat and Hwang (1980) reduced the
support sizes and found minimal supports for many BIB designs.
When dealing with BIB designs one need only use (v,k,2)
trades, but (v,k,t) trades have similar applications for
altering the support of t-designs (t ≥ 3) .

3. Generating a block of maximal multiplicity in the com- plete BIB(v , 3 , v − 2) design

We shall be concerned here with the complete BIB(v , 3 , v − 2)
design. The object is to find disjoint trades which when
added to this initial design yield a design in which a block

occurs a maximal number of times. The support of a design with parameters as above and with a block of maximal multiplicity is also examined.

Proposition 3.1. If a BIB design with parameters $v (\geq 7)$, $k = 3$ and $\lambda = v - 2$ has a block repeated $v - 2$ times then its support satisfies $b^* \leq \binom{v}{3} - 4(v-3)$.

Proof: Let us assume the existence of a design d with parameters $v \geq 7$, $k = 3$ and $\lambda = v - 2$ and having a block repeated a maximal number of times. We can assume that the repeated block in question is (123) or else we can relabel the varieties.

For an arbitrary block γ we write

$$c_i = \text{number of blocks in } d \cdot \text{intersecting}$$

$$\gamma \text{ in } i \text{ varieties } (i = 0,1,2,3).$$

Then, given γ, the following (incidence-adjacency relations) are well-known equations relating the parameters of the design to the symbols just introduced

$$\left. \begin{array}{l} c_0 + c_1 + c_2 + c_3 = \binom{v}{3} \\[2mm] c_1 + 2c_2 + 3c_3 = 3 \dfrac{(v-1)(v-2)}{2} \\[2mm] c_2 + 3c_3 = 3(v-2) \end{array} \right\} \qquad (3.1)$$

Taking $\gamma = (123)$ we have $c_3 = v - 2$ and the equations (3.1) give further

$$c_0 = \binom{v}{3} - \frac{3}{2}(v-2)(v-3) - (v-2)$$

$$c_1 = 3\,\frac{(v-2)(v-3)}{2}$$

$$c_2 = 0 \qquad\qquad\qquad\qquad (3.2)$$

$$c_3 = v - 2$$

Since (123) is repeated $v - 2$ times in d the pairs (12), (23) and (13) will clearly appear only in (123). So the blocks $\{(12x)\,,(23y)\,,(13z) : x\,,y\,,z \notin \{1\,,2\,,3\}\}$ must be outside the support of d. There are $3(v-3)$ such blocks. From (3.2) $c_0 = \binom{v}{3} - \frac{3}{2}(v-2)(v-3) - (v-2) = \binom{v-3}{3} - (v-3)$. So there are $\binom{v-3}{3} - (v-3)$ blocks in d disjoint from (123). The total number of blocks disjoint from (123) is clearly $\binom{v-3}{3}$. There will therefore be at least $(v-3) = \binom{v-3}{3} - c_0$ blocks that are disjoint from (123) and are outside the support of d. We therefore found at least $4(v-3)$ blocks outside the support of d. Thus the support of d is at most $\binom{v}{3} - 4(v-3)$. This ends the proof.

The next proposition asserts the existence and gives a construction of such a design.

<u>Proposition 3.2.</u> There exists a BIB design with parameters $v(\geq 7)$, $k = 3$, $\lambda = v - 2$ in which a block is repeated a maximal number of times and with support $b^* = \binom{v}{3} - 4(v-3)$.

<u>Proof</u>: We start with the complete design $v \geq 7$, $k = 3$, $\lambda = v - 2$, $r = \dfrac{(v-1)(v-2)}{2}$, $b = \binom{v}{3}$. Since we want the block (123) repeated $v - 2$ times, we will construct $v - 3$ trades of size 4 each, containing the block (123) and add them to the complete design. These trades also have the property of reducing the support by 4 everytime one is added. After $v - 3$ such trades have been added, the block (123) will be repeated $v - 2$ times and the resulting support will be $\binom{v}{3} - 4(v-3)$. These are the trades:
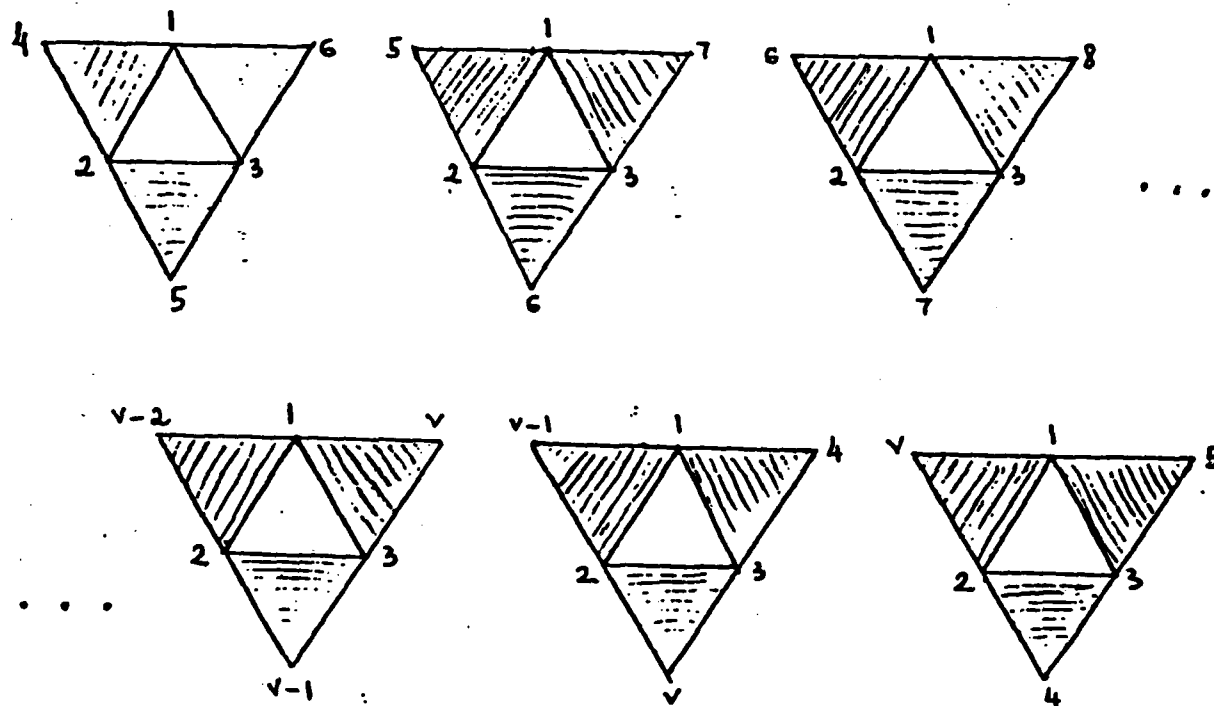
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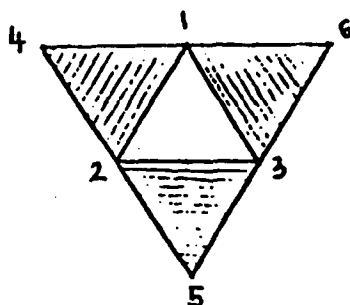
represents the trade:

1 2 3	1 2 4
1 4 6	2 3 5
2 4 5	1 3 6
3 5 6	4 5 6
(white triangles)	(black triangles)

When this trade is added to the complete design the black triangles are subtracted from the support and copies of the white triangles are added. We check that the above $v - 3$ trades contain $4(v-3)$ distinct black triangles.

$(1\,2\,4)$, $(1\,2\,5)$, $(1\,2\,6)$,...,$(1\,2\,v-2)$, $(1\,2\,v-1)$, $(1\,2\,v)$

$(2\,3\,5)$, $(2\,3\,6)$, $(2\,3\,7)$,...,$(2\,3\,v-1)$, $(2\,3\,v)$, $(2\,3\,4)$

$(1\,3\,6)$, $(1\,3\,7)$, $(1\,3\,8)$,...,$(1\,3\,v)$, $(1\,3\,4)$, $(1\,3\,5)$

$(4\,5\,6)$, $(5\,6\,7)$, $(6\,7\,8)$,...,$(v-2,v-1,v)$, $(v-1,v,4)$, $(v\,4\,5)$.

All blocks in the first row are clearly distinct and contain the pair (12). The pair (12) does not occur in any of the rows 2, 3 or 4. A similar argument using pairs (23) and (13)

shows that the blocks in the first three rows are distinct among themselves and from those in row 4. The blocks of row 4 are distinct from those in the first 3 rows since they do not involve symbols 1, 2 or 3. Blocks in row 4 are all distinct among themselves since at each step in row 4 we introduce a new symbol. We do have therefore $4(v-3)$ distinct reductions. So after adding these $v - 3$ trades to the complete design we obtain a design with support

$$b^* = \binom{v}{3} - 4(v-3) .$$

Each of the $v - 3$ trades contains (123) as white triangle. We add therefore $v - 3$ copies of (123) to the already existing block (123) in the complete design. So we end up having $v - 2$ copies of the block (123). But $\lambda = v - 2$, so the block (123) is repeated a maximal number of times in the design obtained this way. This concludes the proof.

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